THE SPHERICAL BERNSTEIN PROBLEM IN S^3

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1. Introduction

The classical Bernstein Theorem, first proved by Sergei Bernstein in 1916, asserts that a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying the minimal surface equation is linear. In particular, the only minimal surfaces in $\mathbb{R}^3$ which are graphs of functions defined on all of $\mathbb{R}^2$ are planes. The Bernstein problem asks whether this holds for minimal graphs of functions $f : \mathbb{R}^{n-1} \to \mathbb{R}$, and was shown to be true if $n \leq 8$ and false if $n \geq 9$ by the collective work of Ennio de Giorgi ([DG65]), Frederick J. Almgren Jr. ([Alm66]), James Simons ([Sim68]), Enrico Bombieri ([BDGG69]), and Enrico Gusti ([BDGG69]). In 1966, Almgren showed the case $n = 5$, using the following key fact.

Theorem 1. *(Lemma 1 in* [Alm66]*): Let $f : S^2 \to S^3$ be an immersion of the two dimensional sphere S^2 as a minimal surface in the unit 3-dimensional sphere $S^3 \subset \mathbb{R}^4$. Then f is an embedding, and there exists $v \in S^3$ such that*

$$f(S^2) = S^3 \cap \{x \in \mathbb{R}^4 : xv = 0\}.$$

Theorem 1 asserts that the only immersed minimal spheres in S^3 are embedded *great spheres*, which are the intersections of S^3 with the hyperplanes $v^\perp = \{w \in \mathbb{R}^4 : \langle v, w \rangle = 0\}$ for $v \in S^3$. This result was also proven independently by Eugenio Calabi in 1967, who showed the following more general result

Theorem 2. *(Lemma 5.4 in* [Cal67]*) Suppose there exists a minimal immersion $X : S^2 \to S^{n-1}$ with image not contained in the intersection of a hyperplane with S^{n-1}. Then n is an odd integer.*

The *Spherical Bernstein Problem*, proposed by Shiing-Sheng Chern in 1970, asks whether Theorem 1 holds for minimal immersions $S^{n-1} \to S^n$ with $n > 3$. In 1984, Per Tomter showed that this is false for all even n (see [Tom84]).

In this paper, we give an exposition of Almgren's proof of Theorem 1, which is intended to be accessible to students with some basic knowledge of differential geometry. Some knowledge of complex analysis will also be helpful for the reader. The proof relies on the fact that the only holomorphic quadratic differential on the Riemann Sphere is identically 0 which, as

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Almgrem and Calabi note, had been used before by Heinz Hopf to prove that a compact two dimensional manifold of genus 0 embedded in $\mathbb{R}^3$ with constant mean curvature must be a standard sphere (see [Hop54]). Our exposition covers the main ideas of the proof without going into too much computational detail.

In Section 2, we recall some basic definitions and facts about minimal surfaces in S^3 and Riemann Surfaces, most notably that great spheres in S^n are totally geodesic (Proposition 3) and that the only holomorphic quadratic differential on S^2 is identically zero (Proposition 4). In Section 3, we work through the proof of Theorem 1. In Section 4, we outline some of the arguments which Calabi uses in [Cal67] to show a stronger version of Theorem 1. This outline is more technical, and can be safely ignored without affecting the reader's understanding of the other sections. In Section 5, we end with some concluding remarks.

2. Preliminaries

2.1. **Minimal Submanifolds.** Throughout this paper, we denote by S^3 the unit sphere $\{x \in \mathbb{R}^4 : |x| = 1\}$ in $\mathbb{R}^4$, which is given the induced metric from the inclusion $S^3 \hookrightarrow \mathbb{R}^4$. We recall the basic definitions of minimal submanifolds, using the conventions in [CM11]. Given a Riemannian manifold N^n of dimension n with Riemannian connection ∇^N, an immersed submanifold $\phi : M^m \to N^n$ of dimension m with the induced metric has an induced Riemannian connection ∇^M given by $\nabla^M_X Y = (\nabla^N_X Y)^T$, where X, Y are tangent vector fields to M. The *second fundamental form* of N is the tensor $A(X, Y) = (\nabla^N_X Y)^\perp$, and the trace on M of A is the *mean curvature tensor* $\mathbf{H}$. In an orthonormal frame $E_1, \ldots, E_n$ on M, the vector field $\mathbf{H}$ is given by

$$\mathbf{H} = \sum_{i=1}^{m} A(E_i, E_i).$$

The submanifold M is *minimal* if $\mathbf{H} = 0$, and *totally geodesic* if $A = 0$. Note a totally geodesic submanifold is necessarily minimal. If M is a hypersurface in N with local unit normal $\mathbf{n}$, then the *shape operator* S is the unique operator on tangent vector fields which satisfies

$$\langle S(V), W\rangle = \langle A(V, W), \mathbf{n}\rangle.$$

The operator S is given by $S(V) = -\nabla_V \mathbf{n}$, and is symmetric. The eigenvalues $k_1, \ldots, k_n$ of S are the *principal curvatures* of M, and are defined up to sign. The mean curvature vector satisfies

$$\mathbf{H} = \left(\sum_{i=1}^{n} k_i\right) \mathbf{n}$$

Thus we see that M is totally geodesic if and only if each k_i vanishes, and M is minimal if and only if $\sum_{i=1}^{n} k_i$ vanishes. The Spherical Bernstein Problem asks for a more general converse to the following observation.

Proposition 3. *Let $n \geq 2$ and let $C \subset S^n$ be an $n-1$-sphere of the form $C = \{x \in S^n | \langle x, v\rangle = 0\}$ for some $v \in S^n$. Then C is a totally geodesic submanifold in S^n.*

Proof. It is sufficient to show that the shape operator S on C is identically 0. Let $\mathbf{n}$ be the unit normal to C. For a vector field V tangent to C, $S(V)$ is given by $-\nabla_V^{S^n}\mathbf{n}$, where ∇^{S^n} is the Riemannian connection of S^n. By definition of C, the normal $\mathbf{n}$ is the restriction of the constant vector field with value v on $\mathbb{R}^{n+1}$ to C, where we note that v is tangent to S^n along C. Since ∇^{S^n} is the tangential projection of the Riemannian connection of $\mathbb{R}^n$, it follows that $\nabla_V^{S^n}\mathbf{n} = 0$, hence S is identically 0 as desired.

□

2.2. **Riemann Surfaces.** A *Riemann Surface* R is a connected Hausdorff space with a covering by open sets $\{U_\alpha\}$ and maps $z_\alpha : U_\alpha \to \mathbb{C}$ which are homeomorphisms onto open subsets of $\mathbb{C}$ such that the transition maps $f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$ are holomorphic. The sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is naturally a Riemann Surface with the charts $f^+ : S^2 \setminus \{\infty\} \to \mathbb{C}, f^- : S^2 \setminus \{0\} \to \mathbb{C}$ given by $f^+(z) = z, f^-(z) = 1/z$. A *quadratic differential* on R is an assignment of a holomorphic function $\varphi_1(z_1)$ to each local coordinate z_1 such that, if z_2 is another local coordinate, then $\varphi_1(z_1) = \varphi_2(z_2)(\frac{dz_2}{dz_1})^2$. We write a quadratic differential on R in the form $\varphi(z)dz^2$.

Proposition 4. *Any quadratic differential Ψ on S^2 is identically* 0.

Proof. In the chart f^+, we write Ψ as $\psi(z)dz^2$. Consider the coordinate $w = \frac{1}{z}$ of the chart f^-. We have $\frac{dw}{dz} = -z^{-2}$, hence for $w \neq 0$ ($z \neq \infty$) we can write Ψ as $\psi(z)\frac{dw^2}{w^4}$. By definition of quadratic differential, the function $\psi(z)\frac{1}{w^4}$ defined for $z \in S^2 \setminus \{\infty\}$ agrees for $z \in S^2 \setminus \{\infty, 0\}$ with the holomorphic function associated by Ψ to the coordinate $w = 1/z$ on $S^2 \setminus \{0\}$, hence it extends continuously to a function on S^2. This implies that $\psi(z)\frac{1}{w^4}$ is bounded and $\psi(z) \to 0$ as $w \to 0$ or equivalently $z \to \infty$. It follows that ψ is a bounded entire function, hence by Liouville's Theorem it is constant. Since $\psi(z)$ tends to 0 as $z \to \infty$, it follows that $\psi(z)$ is 0, thus Ψ is identically zero.

□

3. Proof of Theorem 1

Almgren's proof is divided into four parts. The first three are mostly computational and devoted to establishing some important identities, while the last part applies these identities to construct a quadratic differential on S^2 and then apply Proposition 4. In subsection 3.1, we discuss the main results and proof methods in parts $1-3$ of Almgren's Proof without going into many computational details. In subsection 3.2, we describe the application of Proposition 4 used by Almgren in the last part of his proof and go into more detail.

3.1. **Parts $1-3$ in the proof of Lemma 1 in** [Alm66]**.** We denote by $\langle -, - \rangle$ the standard inner product on $\mathbb{R}^4$ and by ∇ the Riemannian connection on $\mathbb{R}^4$. For notational simplicity, we will often denote $\langle V, W \rangle = VW$. Let $f : S^2 \to S^3$ be a smooth immersion of S^2 as a minimal surface.

Remark In [Alm66], Almgren assumes the stronger condition that the immersion $f : S^2 \to S^3$ is real analytic. As Calabi notes on page 117 of [Cal67], after fixing real analytic structures on S^2, S^3, this can be deduced from minimality of the immersion given only that f is C^3. In particular, this will always hold under our assumptions, where we may consider S^2, S^3 with the real analytic structures obtained from stereographic projection.

We give S^2 the metric induced from f, and denote by A the second fundamental form on $f(S^2)$. In a local coordinate system $\Phi(u,v) : U \subset \mathbb{R}^2 \to S^2$ on S^2, we write $f_u = df(\partial_u), f_v = df(\partial_v)$ and use the subscripts u, v for the higher partial derivatives of f. For instance, we write $f_{uu} = \nabla_{f_u} f_u$ and $f_{uuv} = \nabla_{f_v} \nabla_{f_u} f_u$. By orientability of S^2, we can choose a smooth unit normal to $f(S^2)$ in S^3. In particular, there exists some smooth map $g : S^2 \to S^3$ which is tangent to S^3 along $f(S^2)$ in the sense that $\langle f(x), g(x) \rangle_{\mathbb{R}^4} = 0$ for all $x \in S^2$ and normal to $f(S^2)$ in the sense that $\langle g(x), v \rangle = 0$ for all $v \in T_{f(x)} f(S^2)$, where we view $T_{f(x)} f(S^2) = df(T_x S^2)$ as a subspace of $\mathbb{R}^4$. Let k_1, k_2 be the principal curvatures of $f(S^2)$ with respect to g (recall these are the eigenvalues of the shape operator $V \mapsto -\nabla_V g$). By minimality of the immersion, we have $k_1 + k_2 = 0$. In the local coordinates Φ, the induced metric on S^2 from the immersion f has the form

$$\varphi = f_u f_u du \otimes du + f_u f_v (du \otimes dv + dv \otimes du) + f_v f_v dv \otimes dv$$

The proof proceeds by choosing isothermal coordinates on S^2.

Remark Recall that *isothermal coordinates* about a point p in a Riemannian manifold M are coordinates in which the metric on M has the form

$$\kappa(dx_1^2 + \cdots + dx_n^2)$$

for some smooth function $\kappa > 0$. It is a classical result that such coordinates exist about any point in a two dimensional Riemannian Manifold.

Choose local coordinates $\Phi(u,v) : V \subset \mathbb{R}^2 \to S^2$ about a point $p \in S^2$ which are isothermal with respect to metric φ. We denote $\Phi(V) = U$. In these coordinates φ has the form

$$\kappa(du \otimes du + dv \otimes dv) = f_u f_u du \otimes du + 2 f_u f_v du \otimes dv + f_{vv} dv \otimes dv.$$

This implies that

$$f_u f_u = f_v f_v \tag{1}$$

$$f_u f_v = 0 \tag{2}$$

The main result of Part 1 in the proof of Lemma 1 in [Alm66] is the following two identities

$$k_1k_2 = (f_uf_u)^{-2}[(f_{uu}g)(f_{vv}g) - (f_{uv}g)^2] \tag{3}$$

$$k_1 + k_2 = (f_uf_u)^{-1}[f_{uu}g + f_{vv}g] \tag{4}$$

Almgren shows $(3)-(4)$ using a particular choice of orthonormal coordinates on $\mathbb{R}^4$. We give now an alternative proof. The quantities $k_1k_2, k_1 + k_2$ are respectively the determinant and trace of the shape operator S from the immersion f viewed as a linear operator on each tangent space of S^2. In the isothermal coordinates on U, using (1) we have the following orthonormal frame

$$e_1 = \frac{1}{(f_uf_u)^{1/2}} f_u$$

$$e_2 = \frac{1}{(f_vf_v)^{1/2}} f_v = \frac{1}{(f_uf_u)^{1/2}} f_v$$

By definition of the shape operator, we have $\langle S(e_i), e_j\rangle = \langle A(e_i, e_j), g\rangle$. Thus the matrix for S in the frame e_i is given by

$$\frac{1}{f_uf_u}\begin{bmatrix} f_{uu}g & f_{vu}g \\ f_{uv}g & f_{vv}g \end{bmatrix}$$

Taking determinant and trace yields $(3), (4)$ respectively. Identity (4) is immediately useful for deriving consequences of minimality.

In Parts $2, 3$ of the proof of Lemma 1 in [Alm66], Almgren reaches the following identities, which are valid in the isothermal coordinates Φ on the neighborhood U of $p \in S^2$.

$$(f_uf_u)g_u = -(f_{uu}g)f_u - (f_{uv}g)f_v \tag{5}$$

$$(f_uf_u)g_v = -(f_{uv}g)f_u - (f_{vv})f_v \tag{6}$$

$$f_{uu}g + f_{vv}g = 0 \tag{7}$$

$$f_{uuu}g + f_{uu}g_u + f_{uvv}g + f_{vv}g_u = 0 \tag{8}$$

$$f_{uu}f_u - f_{uv}f \tag{9}$$

$$f_{uv}f_u - f_{vv}f_v = 0 \tag{10}$$

$$f_{uv}f_u + f_{uu}f_v = 0 \tag{11}$$

$$f_{vv}f_u + f_{uv}f_v = 0 \tag{12}$$

$$f_{uu}f_v + f_{vv}f_v = 0 \tag{13}$$

$$f_{uu}f_u + f_{vv}f_u = 0 \tag{14}$$

$$(f_{uu}g)_u + (f_{uv}g)_v = 0 \tag{15}$$

$$(f_{uu}g)_v - (f_{uv}g)_u = 0 \tag{16}$$

The identities $(5), (6)$ are obtained using only that $fg = 0$, $gg = 1$ and g is orthogonal to the tangent spaces of $f(S^2)$. The identities $(7) - (16)$

are obtained by direct computation using (1) $-$ (4) and minimality. The identities $(15),(16)$ will be especially important in the next section.

3.2. **Part** 4 **in the proof of Lemma** 1 **in** [Alm66]**.** We use the same notation as in 3.1. This Section is divided into three lemmas. In Lemma 5, we use the identities $(15),(16)$ to construct a holomorphic function in isothermal coordinates about a point in S^2. We then show this function extends to a quadratic differential on S^2, and use Proposition 4 to show that g is constant in Lemma 6 and that f is an embedding in Lemma 7.

In the isothermal coordinate system $\Phi : V \subset \mathbb{R}^2 \to U$, we define complex parameters $w, \overline{w}$ by

$$w = u + iv, \overline{w} = u - iv$$

Define a complex valued function F on U by $F = f_{uu}g - if_{uv}g$. By minimality, and (7) we have $k_1 + k_2 = 0, f_{uu}g = -f_{vv}g$. From this we derive the following identity

$$|k_1 - k_2|^2 = 4(f_u f_u)^{-2}|F|^2. \tag{17}$$

We have

$$\begin{aligned}
|k_1 - k_2|^2 &= k_1^2 - 2k_1k_2 + k_2^2 \\
&= k_1^2 + 2k_1k_2 + k_2^2 - 4k_1k_2 \\
&= (k_1 + k_2)^2 - 4k_1k_2 \text{ (Note } k_1 + k_2 = 0) \\
&= -4(f_u f_u)^{-2}[(f_{uu}g)(f_{vv}g) - (f_{uv}g)^2] \text{ (by (3))} \\
&= -4(f_u f_u)^{-2}[-(f_{vv}g)^2 - (f_{uv}g)^2] \text{ (by (7))} \\
&= 4(f_u f_u)^{-2}|F|^2
\end{aligned}$$

In particular, $F(q) = 0$ if and only if $k_1(q) = k_2(q)$ for $q \in U$. We next show that F is holomorphic and satisfies a useful identity involving the complex derivatives of f, g.

Remark Recall that for a holomorphic function $h : \mathbb{C} \to \mathbb{C}$, the derivatives of h with respect to the complex variables $w = u + iv, \overline{w} = u - iv$ are given by

$$\frac{\partial h}{\partial w} = \frac{1}{2}(\frac{\partial h}{\partial u} - i\frac{\partial h}{\partial v}), \frac{\partial h}{\partial \overline{w}} = \frac{1}{2}(\frac{\partial h}{\partial u} + i\frac{\partial h}{\partial v})$$

We can use these identities to consider the derivatives of the functions $f, g : S^2 \to S^3$ with respect to the complex parameter $w = u+iv$ on the isothermal coordinate neighborhood U of $p \in S^2$.

Lemma 5. *The function F is holomorphic on U and satisfies*

$$F = -2f_w g_w. \tag{18}$$

Proof. Since f, g are smooth, to show F is holomorphic it is sufficient to show F satisfies the Cauchy-Riemann Equations $(\operatorname{Re} F)_u = (\operatorname{Im} F)_v, (\operatorname{Re} F)_v = -(\operatorname{Im} F)_u$. These equations read

$$(f_{uu}g)_u + (f_{uv}g)_v = 0$$
$$(f_{uu}g)_v - (f_{uv}g)_u = 0.$$

These are exactly (15),(16), hence F is holomorphic. We next derive identity (18). Differentiating $f_u g = f_v g = 0$, we obtain the following identities.

$$f_{uu}g + f_u g_u = 0 \tag{19}$$
$$f_{uv}g + f_u g_v = 0 \tag{20}$$
$$f_{uv}g + f_v g_u = 0 \tag{21}$$
$$f_{vv}g + f_v g_v = 0 \tag{22}$$

From (7) and (19) – (22) we have

$$\begin{aligned} f_w g_w &= [2^{-1}(f_u - if_v)][2^{-1}(g_u - ig_v)] \\ &= 4^{-1}[f_u g_u - if_v g_u - ig_v f_u - f_v g_v] \\ &= 4^{-1}[-2f_{uu}g + 2if_{uv}g] \\ &= -2^{-1}F \end{aligned}$$

which proves (18). □

We can now construct a quadratic differential on S^2 which agrees with $F(dw)^2$ in the coordinate neighborhood U of p. Suppose $\Phi'(x, y) : V' \subset \mathbb{C} \to S^2$ is a complex coordinate neighborhood of p with $U' = \Phi'(V')$, which is compatible with $\Phi : V \to S^2$ in the sense that $\Phi' \circ \Phi^{-1}$ is holomorphic where defined on the appropiate intersections. Then the functions $u(x, y), v(x, y)$ satisfy the Cauchy-Riemann Equations $\frac{du}{dx} = \frac{dv}{dy}, \frac{du}{dy} = -\frac{dv}{dx}$, so from the chain rule and that u, v are isothermal we obtain

$$\begin{aligned} f_x f_y &= (f_u \frac{du}{dx} + f_v \frac{dv}{dx})(f_u \frac{du}{dy} + f_v \frac{dv}{dy}) \\ &= f_u f_u (\frac{du}{dx}\frac{du}{dy}) + f_v f_v (\frac{dv}{dx}\frac{dv}{dy}) \\ &= f_u f_u (-\frac{du}{dx}\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx}) = 0 \\ f_x f_x &= (f_u \frac{du}{dx} + f_v \frac{dv}{dx})(f_u \frac{du}{dx} + f_v \frac{dv}{dx}) \\ &= f_u f_u (\frac{du}{dx})^2 + f_v f_v (\frac{dv}{dx})^2 \\ &= f_v f_v (\frac{dv}{dy})^2 + f_u f_u (\frac{du}{dy})^2 \\ &= (f_u \frac{du}{dy} + f_v \frac{dv}{dy})(f_u \frac{du}{dy} + f_v \frac{dv}{dy}) = f_y f_y \end{aligned}$$

so the coordinates Φ' are isothermal, and we can define the holomorphic function $G(z = x+iy)$ in the coordinates Φ' analogously to F. By doing this in each compatible complex coordinate we obtain a function assigned to each complex coordinate. It remains to check the formula $G(z) = F(w)(\frac{dw}{dz})^2$. From (17) and the chain rule we have

$$G(z) = -2f_z g_z = -2(f_w \frac{dw}{dz})(g_w \frac{dw}{dz}) = -2f_w g_w (\frac{dw}{dz})^2 = F(w)(\frac{dw}{dz})^2$$

Thus we obtain the desired quadratic differential on S^2, which we denote by Ψ.

Lemma 6. *The unit normal g is constant. In particular, there exists $v \in S^3$ such that $f(S^2) = S^3 \cap \{x : xv = 0\}$.*

Proof. By Proposition 4, we conclude that Ψ is identically 0, hence $F = 0$ and from (17) we see that $k_1 = k_2$. Since $k_1 + k_2 = 0$, this implies that $k_1 = k_2 = 0$. To show that g is constant, is sufficient to show that the inner products of the derivatives of g with f, g, f_u, f_v vanish, as these form an orthogonal basis for $\mathbb{R}^4$ at each point of $f(S^2)$. Since $F = 0$, we have $f_{vv}g = f_{uu}g = f_{uv}g = 0$, hence $(19), (20), (21), (22)$ imply that $f_u g_u = f_u g_v = f_v g_u = f_v g_v = 0$. Since $fg = 0$, the chain rule yields

$$f_u g + f g_u = f_v g + f g_v = 0 \tag{23}$$

Note that f_u, f_v are tangent to $f(S^2)$, which implies that $f_u g = f_v g = 0$, hence by (23) we have $fg_u = fg_v = 0$. Since $gg = 1$, we have $2g_u g = 2g_v g = 0$. It follows that g is constant, hence since $fg = 0$ we obtain that $f(S^2)$ is the intersection of S^3 with a hyperplane in $\mathbb{R}^4$. In particular, there exists $v \in S^3$ such that $f(S^2) = S^3 \cap \{x : xv = 0\}$. □

We are now able to complete the proof of Theorem 1.

Lemma 7. *The minimal immersion $f : S^2 \to S^3$ is an embedding*

Proof. We claim the restriction $f : S^2 \to f(S^2)$ is a covering map. Let $p \in f(S^2)$. Since S^2 is compact, the preimage $f^{-1}(p)$ is compact. Since f is an immersion, it is a local embedding, thus the preimage $f^{-1}(p)$ is discrete. It follows that $f^{-1}(p)$ is finite. Since f is a local embedding, for each $q \in f^{-1}(p)$ there exists a neighborhood $q \in U_q \subset S^2$ such that f restricts to diffeomorphism $f : U_q \to f(U_q)$. Then the neighborhood $\cap_{q \in f^{-1}(p)} f(U_q)$ of p is evenly covered by f, thus $f : S^2 \to f(S^2)$ is a covering map. By lemma 6, $f(S^2) = S^3 \cap \{x : xv = 0\}$ is homeomorphic to S^2, which is simply connected and hence has no nontrivial covers. It follows that f is injective and a homeomorphism onto its image, hence f is an embedding. □

4. An outline of Calabi's proof in [Cal67]

Calabi's proof of Theorem 2 is more technical than Almgren's, but ultimately relies on a similar application of Proposition 4. Let $X : S^2 \to S^{n-1} \subset \mathbb{R}^n$ be a minimal immersion with image not contained in the intersection of any hyperplane in $\mathbb{R}^n$ with S^{n-1}. We give S^2 the induced metric from the immersion X. In isothermal coordinates $w = u + iv$ on $U \subset S^2$, we write the induced metric on S^2 from X as $2F(w, \overline{w})(du^2 + dv^2)$.

Remark For a function h of a complex parameter $w = u + iv$, we often write h as a function $h(w, \overline{w})$ of $w, \overline{w}$. As in Section 3.1, we take derivatives with respect to $w, \overline{w}$ using the formulas $\frac{\partial h}{\partial z} = \frac{1}{2}(\frac{\partial h}{\partial u} - i\frac{\partial h}{\partial v})$, $\frac{\partial h}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial h}{\partial u} + i\frac{\partial h}{\partial v})$.

Calabi's proof is based on considering Sections of certain vector bundles on S^2, which we briefly describe in this Section. For integers k, l, we denote by $E^{k,l}$ the complex line bundle on S^2 whose elements are equivalence classes of quadruples (U, w, q, v) where U is an open domain in S^2, w is an isothermal parameter on U, $q \in U, v \in \mathbb{C}$ by the relation identifying $(U, w, p, v), (U', w', p', v')$ if and only if the following hold

$$p = p' \tag{24}$$

$$v' = v(\frac{\partial w}{\partial w'}(q))^k \overline{(\frac{\partial w}{\partial w'}(q))}^l \tag{25}$$

For instance, $E^{-1,0} \oplus E^{-1,0}$ is the complexified tangent bundle $TS^2 \otimes_{\mathbb{R}} \mathbb{C}$, while a Riemannian metric on S^2 is a section of $E^{1,1}$. Sections of the bundle $E^{k,l}$ can be identified with functions $v = f(p)$ satisfying the transformation rule (25) in isothermal coordinates. In particular, a complex quadratic differential is a section of $E^{2,0}$. There is a Levi-Civita connection ∇ on the bundle $\oplus_{k,l} E^{k,l}$ written uniquely as a sum $\nabla' + \nabla''$ where, for a section of $E^{k,l}$ written in local isothermal coordinates as a complex valued function $f(w, \overline{w})$ we have

$$\nabla' f = (\frac{\partial f(w, \overline{w})}{\partial \overline{w}} - p\frac{\partial \log F(w, \overline{w})}{\partial w} f(w, \overline{w}))$$
$$\nabla'' f = (\frac{\partial f(w, \overline{w})}{\partial \overline{w}} - q\frac{\partial \log F(w, \overline{w})}{\partial \overline{w}} f(w, \overline{w}))$$

Recall that the Grassman Algebra $\lambda(V)$ of a vector space V is the exterior algebra spanned by the exterior products $v_1 \wedge \cdots \wedge v_k, v_i \in V$. A *k-vector* in $\lambda(V)$ is an exterior product of k vectors in V. Calabi's proof of Theorem 2 relies on two symmetric bilinear forms defined on $\mathbb{C}^n, \lambda(\mathbb{C}^n)$ respectively. For $z' = (z_1', \ldots, z_n'), z'' = (z_1'', \ldots, z_n'') \in \mathbb{C}^n$ and $Z = z_1 \wedge \cdots \wedge z_k, W = w_1 \wedge \cdots \wedge w_k$ two k-vectors in $\lambda(\mathbb{C}^n)$, we set

$$\langle z', z'' \rangle = \sum_{\alpha=1}^{n} z_\alpha' z_\alpha'' \tag{26}$$

$$\langle Z, W \rangle = \det_{1 \leq \alpha, \beta \leq p} (\langle z_\alpha, w_\beta \rangle) \tag{27}$$

These forms give norms on $\mathbb{C}^n, \lambda(\mathbb{C}^n)$ by setting $|z'|^2 = \langle z, \overline{z} \rangle, |Z|^2 = \langle Z, \overline{Z} \rangle$, where the conjugates are given by $\overline{z}' = (\overline{z}_1', \ldots, \overline{z}_n'), \overline{Z} = \overline{z}_1 \wedge \cdots \wedge \overline{z}_k$.

We write $(\nabla')^p(\nabla'')^q X, (\nabla'')^p(\nabla')^q X$ respectively for the vector fields on S^2 which in a local complex parameter $w = u+iv$ are given by $\frac{\partial^{p+q}X}{\partial w^p \partial \overline{w}^q}, \frac{\partial^{p+q}X}{\partial \overline{w}^p \partial w^q}$ and adopt the convention that $\frac{\partial^{p+q}X}{\partial w^p \partial \overline{w}^q} = \frac{\partial^{p+q}X}{\partial \overline{w}^p \partial w^q} = 0$ if $p = q = 0$. From the formulas for the derivative with respect to a complex variable, the derivative $\frac{\partial^{p+q}X}{\partial w^p \partial \overline{w}^q}$ makes sense as an element of $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n$, hence we can consider the complex span of subsets of these derivatives. By an analogous argument to the proof of Proposition 4, one can show that any holomorphic section of $E^{k,0}$ is identically 0 for any $k > 0$. Using this fact, Calabi shows that the complex subspace of $\mathbb{C}^n$ spanned by the derivatives $(\nabla')^p X$ is totally isotropic with respect to the dot product defined in (26) in the sense that, for any $p, q \geq 0$ with $p + q \geq 1$, we have $\langle (\nabla')^p X, (\nabla')^q X \rangle = 0$. This is then used by Calabi to compute the dimension of the sphere S^{n-1}.

For each $k > 0$, we define $T'_k X$ as the following exterior product

$$T'_k X = \nabla' X \wedge (\nabla')^2 X \wedge \cdots \wedge (\nabla')^k X.$$

This product should be thought of as a section of the bundle $E^{k_2,0} \otimes_{\mathbb{C}} \lambda(\mathbb{C}^n)$, where $k_2 = \frac{1}{2}k(k+1)$, and will vanish if and only the vectors $\nabla' X, \ldots, (\nabla')^k X$ are linearly dependent in $\mathbb{C}^n$. Calabi shows that the vanishing of $T'_k X$ is directly related to the value of n. In particular, the precise statement of Theorem 2 is as follows

Theorem 8. *(Lemma 5.4 in* [Cal67]*) Suppose there exists a minimal immersion $X : S^2 \to S^{n-1}$ with image not contained in the intersection of any hyperplane in $\mathbb{R}^n$ with S^{n-1}, then $n = 2m+1$, where m is the largest positive integer such that $T'_m X$ is not identically* 0.

Calabi's proof of Theorem 8 is separated into two steps, one showing $n \geq 2m+1$, and the next showing $n \leq 2m+1$. The first step uses the fact that the space spanned by the derivatives $(\nabla')^k X, (\nabla'')^k X$ is totally isotropic to show that the $2m+1$-vector

$$X \wedge (\nabla' X \wedge (\nabla')^2 X \wedge \cdots \wedge (\nabla')^m X) \wedge (\nabla'' X \wedge (\nabla'')^2 X \wedge \cdots \wedge (\nabla'')^m X)$$

has nonzero norm, which implies that $\mathbb{C}^n$ contains $2m+1$-linearly independent vectors and hence $n \geq 2m+1$. The second step is less direct but ultimately relies on the fact that, since X does not have image contained in the intersection of a hyperplane with S^{n-1} and is analytic (recall the latter can be deduced from minimality), the complex span of the derivatives of X of sufficiently high orders will be all of $\mathbb{C}^n$.

5. Concluding Remarks

Both Almgren and Calabi's proofs show the utility of tools from complex analysis in Riemannian geometry. Together with Hopf's original work, they suggest a general procedure for working with immersions of S^2 that are minimal or have constant mean curvature: Use some derivatives of the immersion to construct some analogue of a quadratic differential on S^2, and then use the vanishing of such a differential to reach some contradiction.

This approach is very specific to the sphere S^2, and is not immediately applicable in higher dimensions where isothermal coordinates may not exist.

Theorems 1 and 2 are results of independent interest, but were originally considered by Almgren and Calabi for the purpose of understanding the *tangent cones* of three dimensional minimal submanifolds of $\mathbb{R}^n$ that have singularities. These tangent cones, after taking some cutoff, are minimal submanifolds of $\mathbb{R}^n$ with boundary a sphere S^2 immersed minimally in S^{n-1} (see the Introductions of [Alm66],[Cal67]). Theorems 1 and 2 then apply, and Theorem 1 is used by Almgren to show that any such tangent cone in $\mathbb{R}^4$ is a hyperplane, which extends Bernstein's Theorem to a function $f : \mathbb{R}^4 \to \mathbb{R}$ by work of De Giorgi (see [DG65]) and Fleming ([Fle62]). These connections between the classical Bernstein problem and minimal immersions of spheres are motivation for considering the Spherical Bernstein Problem, and led to rich areas of research over the twentieth century.

References

[Alm66] F. J. Almgren. Some interior regularity theorems for minimal surfaces and an extension of bernstein's theorem. *Annals of Mathematics*, 84(2):277–292, 1966.

[BDGG69] E Bombieri, E De Giorgi, and E Giusti. Minimal cones and the bernstein problem. *Ennio De Giorgi*, page 291, 1969.

[Cal67] Eugenio Calabi. Minimal immersions of surfaces in euclidean spheres. *Journal of Differential Geometry*, 1(1-2):111–125, 1967.

[CM11] Tobias H Colding and William P Minicozzi. *A course in minimal surfaces*, volume 121. American Mathematical Soc., 2011.

[DG65] E De Giorgi. Una extensione del theorema di bernstein. *Ann. Scuola Norm. Sup Pisa*, pages 79–80, 1965.

[Fle62] Wendell H Fleming. On the oriented plateau problem. *Rendiconti del Circolo Matematico di Palermo*, 11(1):69–90, 1962.

[Gar87] Frederick P Gardiner. Teichmuller theory and quadratic differentials. *Pure Appl. Math.*, 1987.

[Hop54] Heinz Hopf. Lectures on differential geometry in the large. *Reprinted in Lecture Notes in Math.*, 1000, 1954.

[Sim68] James Simons. Minimal varieties in riemannian manifolds. *Annals of Mathematics*, pages 62–105, 1968.

[Tom84] Per Tomter. The spherical bernstein problem in even dimensions and related problems. *Preprint series: Pure mathematics http://urn. nb. no/URN: NBN: no-8076*, 1984.